Asymptotic behavior of the density of states on a random lattice

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Abstract. We study the diffusion of a particle on a random lattice with fluctuating local connectivity of average value q. This model is a basic description of relaxation processes in random media with geometrical defects. We analyze here the asymptotic behavior of the eigenvalue distribution for the Laplacian operator. We found that the localized states outside the mobility band and observed by Biroli and Monasson, in a previous numerical analysis [1], are described by saddle point solutions that breaks the rotational symmetry of the main action in the real space. The density of states is characterized asymptotically by a series of peaks with periodicity $1/q$.

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Diffusion on random graphs can be a useful problem for studying relaxation processes in glassy systems in general. Usually, the disorder arises from a random potential, impurities, but a random geometry can also play this role [2]. One can visualize a diffusion of a particle on a random graph as the relaxation of a disordered system out of equilibrium on a complicate energy landscape. For example, this relaxation in random Ising magnets can be identified with diffusion on the vertices of a hypercube in the configuration space, and the edges correspond to the energy paths that connect one configuration to another [3]. Numerical simulations [3, 4] in Ising spin glasses of the order parameter $q(t) = [< S_i(t)S_i(0) >]_{av}$, where $< \ldots >$ is the thermal average and $[\ldots]_{av}$ the average over disorder, show that this quantity follows a Kohlrausch law similar to a "stretched" exponential $\exp(-(t/\tau)^{\beta})$ with $1/3 \le \beta \le 1$ in the region just above the spin glass phase. This kind of non-exponential relaxation is typical of many glassy systems [2, 5, 6], unlike the usual exponential behavior with only one relaxation time. The coefficient β varies with temperature from 1 in the high temperature phase, to 1/3 at the glassy transition. In between, there seems to be a phase of localized states, or Griffiths phase, where β is in between from 1 to 1/3. The value 1/3 seems to be universal in many experimental systems and has been observed in spin glass $Eu_{0.4}Sr_{0.6}S$ [7], at the glass transition of the crystalline fast ion conductor Na β-alumina [8], or molten salts $CaK(NO_3)$ [9]. The question about the universality of the Kohlrausch law has been raised, and exact dynamics on ultra-metric spaces [10] show that it depends on the scaling of the barriers with the distance between different states. It could be algebraic (linear dependence with the distance) or Kohlrausch like (logarithm dependence). Random graph models can also be a useful tool to study propagation of sound waves in granular and random media [11, 12], since diffusion, quasi-diffusion and localization of sound waves are similar to diffusion on a random network, the edges being the connections between the particles making up the medium. Frequency response [12] for different amplitudes shows a region of coherent propagation plus a quasi-diffusive regime. Numerical and simplified models [13] of a granular medium for small vibration amplitudes show modes that are extended and localized in space, like a particle moving on a random medium.

In order to simulate the glassy systems discussed above with short-range interactions and also percolation problems, Viana, Rodgers and Bray (VRB) studied a simple model of diffusion based on sparse random matrix [14, 15]. Their model has the property to have long-range interactions with dilute coordination numbers, so that in average the coordination number is finite and we may expect to have a closer view of some experimental material. The spectrum of eigenvalues (all positive) spreads over a continuous band, and for the particular case of the Cayley tree model, are bounded between mobility edges, with a gap from below (see also [16]). Recently, Biroli and Monasson (BM) [1] studied the same model numerically, and showed that localization

effects arise outside the extended region, with peaks at some regular intervals where the inverse partition number is high, showing that these peaks are resonance for localization. The existence of a Griffiths region was analyzed by Rodgers and Bray [17] who found a tail distribution for large eigenvalues outside the mobility band (with a random matrix containing only elements -1, 0 and 1) but there is no peak structure in their analysis contrary to BM's numerical result. Their saddle point solution is invariant by rotation in both replica and real spaces and they do not discuss the possibility of the rotational symmetry breaking in at least one of these spaces. We will analyze in this paper the possibility of a real space symmetry breaking for the site dependent fields, unlike the method of self-consistent field equations developped in [17]. It is clear that in order to describe the localized regime, we have to take account of this kind of solutions. Our motivation is therefore to find the asymptotic solutions in the region of the density of states outside the mobility band for the VRB's model. The original model of relaxation in a geometrical disordered system consists of N points connected randomly by bonds $M_{ij} = -1/q$ with a probability equal to q/N and $M_{ij} = 0$ else. This is an infinite range model, but the average coordination number q is finite and we may expect that the dilute exchange interactions make the model similar to a short range model [18]. We follow the references [1] and [15] for the notations. On a random lattice, a particle performs a random walk from one point to another. Let $c_i(t)$ be the probability for the particle to be on the site i at time t. Then the master equations describing the time evolution of these amplitudes are

$$\frac{dc_i}{dt} = -\sum_j M_{ij} c_j \tag{1}$$

where the probability for the symmetric elements M_{ij} is

$$P(M_{ij}) = \frac{q}{N}\delta(M_{ij} + \frac{1}{q}) + (1 - \frac{q}{N})\delta(M_{ij}),$$

for $i \neq j$. The diagonal elements are equal to $M_{ii} = -\sum_{j\neq i} M_{ij}$. This ensures that the eigenvalues of the matrix M are all positive since the quadratic form

$$\sum_{i,j} M_{ij} x_i x_j = -1/2 \sum_{i,j} M_{ij}(x_i - x_j)^2$$

is always positive [15]. Let λ_i be an eigenvalue of $\mathbf{M}$, with $\mathbf{V}_i$ the corresponding eigenvector, then the general solution of equation (1) is

$$c_i(t) = \sum_{j,k} (\mathbf{V}_j)_i (\mathbf{V}_j)_k c_k(0) \exp(-\lambda_j t). \tag{2}$$

If the particle is on the site m at $t = 0$, i.e. $c_i(0) = \delta_{i,m}$, then the probability that after a time t the particle is found on the same site is equal to $c_m(t)$, and we can define an average probability of return to the origin at time t after summing up over all the possible origin points:

$$
\begin{aligned}
f(t) = & \left[\frac{1}{N}\sum_m c_m(t)\right]_{av} = \left[\frac{1}{N}\sum_{m,j}(\mathbf{V}_j)_m(\mathbf{V}_j)_m \exp(-\lambda_j t)\right]_{av} \\
= & \left[\frac{1}{N}\sum_j \exp(-\lambda_j t)\right]_{av} = \int_0^\infty P(\lambda)\exp(-\lambda t)d\lambda, \qquad (3)
\end{aligned}
$$

where $[\ldots]_{av}$ is the average over the link configurations. $P(\lambda)$ is the probability distribution function for the eigenvalues of the symmetrical matrix $\mathbf{M}$. In figure 1 we have solved numerically P by diagonalizing 4500 different samples of random matrices $\mathbf{M}$ with the parameters $N = 800$ and $q = 20$ (see also [1]). Then we made an histogram of all the different eigenvalues found and normalized the distribution. Following the results from the partition number of reference [1], the distribution presents a mobility band between approximatively $\lambda = 0.5$ and $\lambda = 1.7$ plus localization edges outside this band and containing peaks. The distribution $P(\lambda)$ can be expressed as a functional of replicated fields ϕ_i^α. Indeed, we can write

$$
\begin{aligned}
P(\lambda) = & \left[\frac{1}{N}\sum_i \delta(\lambda - \lambda_i)\right]_{av} \\
= & \left[\frac{1}{N}\operatorname{tr}\delta(\lambda - \mathbf{M})\right]_{av} = \left[-\mathrm{Im}\frac{1}{\pi N}\operatorname{tr}G(\lambda + i\epsilon)\right]_{av},
\end{aligned}
$$

where we introduce the Green function $G(\lambda + i\epsilon) = 1/(\lambda + i\epsilon - \mathbf{M})$. More precisely:

$$
\begin{aligned}
P(\lambda) = & \left[-\mathrm{Im}\frac{1}{\pi N}\frac{\partial}{\partial\lambda}\operatorname{tr}\log(\lambda + i\epsilon - \mathbf{M})\right]_{av} \\
= & \left[-\mathrm{Im}\frac{1}{\pi N}\frac{\partial}{\partial\lambda}\log\det(\lambda + i\epsilon - \mathbf{M})\right]_{av}.
\end{aligned}
$$

The determinant can be rewritten as a Gaussian integral over fields ϕ_i, $i = 1, \ldots, N$:

$$
\begin{aligned}
P(\lambda) = & \left[\mathrm{Im}\frac{2}{\pi N}\frac{\partial}{\partial\lambda}\log\int\prod_{i=1}^N d\phi_i \exp\left(\frac{i}{2}\sum_i(\lambda + i\epsilon)\phi_i^2\right)\right. \\
& \left.\exp\left(-\frac{i}{2}\sum_{i,j}\phi_i M_{ij}\phi_j\right)\right]_{av}. \qquad (4)
\end{aligned}
$$

The average over the link configurations is performed by using replicated fields ϕ_i^α, and we obtain (see [15] for the details):

$$\begin{aligned}
P(\lambda) = & \left[\mathrm{Im} \frac{2}{\pi N} \frac{\partial}{\partial \lambda} \log Z \right]_{av} = \lim_{n\to 0} \mathrm{Im} \frac{2}{\pi n N} \frac{\partial}{\partial \lambda} \left[Z^n \right]_{av} \\
= & \lim_{n\to 0} \mathrm{Im} \frac{2}{\pi n N} \frac{\partial}{\partial \lambda} \int \prod_{i,\alpha} d\phi_i^\alpha \exp \left(\frac{i}{2} \sum_{i,\alpha} (\lambda + i\epsilon) \phi_i^{\alpha 2} \right) \\
& \exp \left(\frac{1}{2} \sum_{i,j} \log \left[1 - \frac{q}{N} + \frac{q}{N} \exp - \frac{i}{2q} \sum_\alpha (\phi_i^\alpha - \phi_j^\alpha)^2 \right] \right) \\
= & \lim_{n\to 0} \mathrm{Im} \frac{2}{\pi n N} \frac{\partial}{\partial \lambda} \int \prod_{i,\alpha} d\phi_i^\alpha \exp S(\lambda, \{\phi_i^\alpha\}). \qquad (5)
\end{aligned}$$

Now we want to study the asymptotic behavior of the distribution P by evaluating the saddle points of S in the limit $\lambda \to \infty$, while we keep N large but fixed first. It seems very similar to studying the saddle point solutions of the extensive function S in the large N limit, but the difference is that we are studying the asymptotics of the distribution which does not describe the mobility part of the curve, so the corresponding saddle points are different. Moreover we are looking for solutions which can break the rotational space symmetry, since we are interested in the localized regime where strong or low connectivity may contribute to the distribution. A particle localized in one region would favor the amplitudes of the sites inside this region. The extrema of the functional S in equation (5) are given by the set of equations

$$(\lambda + i\epsilon)\phi_i^\alpha = \frac{1}{N} \sum_j \frac{\left(\phi_i^\alpha - \phi_j^\alpha\right) \exp\left(-\frac{i}{2q}\sum_\beta (\phi_i^\beta - \phi_j^\beta)^2\right)}{1 - q/N + q/N \exp\left(-\frac{i}{2q}\sum_\beta (\phi_i^\beta - \phi_j^\beta)^2\right)}. \qquad (6)$$

The simplest solution is when all fields have the same value on every site $\phi_i^\alpha = \phi^\alpha$, which directly leads to $\phi_i^\alpha = \phi^\alpha = 0$. This gives a Dirac function centered at $\lambda = 1$ and this is not an asymptotic solution for our problem. The next step is to take $\phi_i^\alpha = \phi^\alpha$ everywhere except on one site i_0 where $\phi_{i_0}^\alpha \neq \phi^\alpha$. We obtain 2 equations (up to the order $1/N$):

$$\begin{aligned}
(\lambda + i\epsilon)\phi_{i_0}^\alpha &= \frac{N-1}{N} \left(\phi_{i_0}^\alpha - \phi^\alpha\right) \exp\left(-\frac{i}{2q}\sum_\beta (\phi_{i_0}^\beta - \phi^\beta)^2\right), \\
(\lambda + i\epsilon)\phi^\alpha &= \frac{1}{N} \left(\phi^\alpha - \phi_{i_0}^\alpha\right) \exp\left(-\frac{i}{2q}\sum_\beta (\phi^\beta - \phi_{i_0}^\beta)^2\right). \qquad (7)
\end{aligned}$$

In the large N limit, only the field $\phi_{i_0}^\alpha$ does not vanish ($\phi^\alpha = -\phi_{i_0}^\alpha/(N-1)$), with the length of the vector $(\phi_{i_0}^1, \ldots, \phi_{i_0}^n)$ satisfying the equation:

$$\phi_m^2 = \sum_\alpha \phi_{i_0}^{\alpha 2} = 2iq \log(\lambda + i\epsilon) + 4\pi q m, \text{ and } \phi_{i\neq i_0}^\beta \simeq -\frac{\phi_{i_0}^\beta}{N}, \quad (8)$$

where m is an integer. There is therefore an infinite set of non zero solutions in the complex plane. In the following we choose the site located at $i_0 = 1$ for simplification, since the solution is invariant for the $N-1$ other sites. For each integer m, the number of all possible vectors with module ϕ_m^2 is equal to $N2\pi^n/\Gamma(n/2) \sim Nn$ in the limit $n \to 0$. Given the solution (8), the value of the action is equal to

$$S_m = -q(\lambda + i\epsilon)\log(\lambda + i\epsilon) + q(\lambda + i\epsilon) - q + 2i\pi q(\lambda + i\epsilon)m \quad (9)$$

then P falls exponentially with $\lambda \log(\lambda)$, and the corrections to the exponential are given by computing the fluctuations around the saddle points. Unfortunately the matrix of the second derivatives of S at the site $i_0 = 1$ has many zero eigenvalues and we need to consider the next order. Indeed, we have

$$\begin{aligned} \frac{\partial^2 S}{\partial\phi_1^\alpha \partial\phi_1^\beta} &= -\frac{\lambda + i\epsilon}{q}\phi_1^\alpha\phi_1^\beta, \quad \frac{\partial^2 S}{\partial\phi_i^\alpha \partial\phi_j^\beta} = \frac{i}{N}\delta_{\alpha\beta}, \quad i \neq j, \\ \frac{\partial^2 S}{\partial\phi_i^\alpha \partial\phi_i^\beta} &= i(\lambda + i\epsilon - 1)\delta_{\alpha\beta}, \quad i, j \neq 1, \end{aligned} \quad (10)$$

The first matrix in (10) has $(n-1)$ zero eigenvalues, and we therefore have to take account of the third derivatives at the site i_0:

$$\begin{aligned} \frac{\partial^3 S}{\partial\phi_1^\alpha \partial\phi_1^\beta \partial\phi_1^\gamma} &= -\frac{\lambda + i\epsilon}{q}(\phi_1^\alpha\delta_{\beta\gamma} + \phi_1^\beta\delta_{\alpha\gamma} + \phi_1^\gamma\delta_{\alpha\beta}) \\ &\quad + \frac{i(\lambda + i\epsilon)}{q^2}\phi_1^\alpha\phi_1^\beta\phi_1^\gamma. \end{aligned} \quad (11)$$

The fluctuations around the saddle point solutions ϕ_m lead to the following integrals for main contribution at large N:

$$\begin{aligned} I_m(\lambda) = \int \prod_{i,\alpha} dx_i^\alpha \exp\Bigg(&-\frac{\lambda + i\epsilon}{2q}\phi_1^\alpha\phi_1^\beta x_1^\alpha x_1^\beta + \frac{i}{2}(\lambda + i\epsilon - 1)\sum_{i\neq 1} x_i^\alpha x_i^\alpha \\ &+ \frac{i}{2N}\sum_{i\neq j} x_i^\alpha x_j^\alpha - \frac{\lambda + i\epsilon}{2q}\phi_1^\alpha x_1^\alpha x_1^\beta x_1^\beta + \frac{i(\lambda + i\epsilon)}{6q^2}\phi_1^\alpha\phi_1^\beta\phi_1^\gamma x_1^\alpha x_1^\beta x_1^\gamma + \ldots \Bigg). \end{aligned}$$

The Gaussian integration over the variables x_i^α gives $(2\pi/i(\lambda+i\epsilon))^{n(N-1)/2}$. In the following we will set $\epsilon = 0$, since it is not essential for the remaining calculation. It is useful to work in the basis where the quadratic terms are diagonal. We first set $\phi^\alpha = \phi_m \varphi^\alpha$ where $(\varphi^1, \ldots, \varphi^n)$ is a real unit vector, and ϕ_m a square root of (6). The matrix $\varphi_\alpha \varphi_\beta$ has one eigenvalue unity with eigenvector $\mathbf{V}^1 = (\varphi^1, \ldots, \varphi^n)$ and $n-1$ eigenvalues zero with eigenvectors $\mathbf{V}^\beta, \beta = 2, ..n$ satisfying $\sum_\alpha \varphi_\alpha V_\alpha^\beta = 0$. We then use new variables $y_\alpha = \sum_\beta V_\beta^\alpha x_1^\beta$ that diagonalize the quadratic terms. The matrix V_α^β satisfies ${}^t\mathbf{V} = \mathbf{V}^{-1}$, so that $x_1^\alpha = \sum_\beta V_\alpha^\beta y_\beta$. After some algebra, I_m can be finally written as

$$I_m(\lambda) = \left(\frac{2\pi}{i\lambda}\right)^{n(N-1)/2} \int \prod_\alpha dy_\alpha \exp\left(-\frac{\lambda}{2q}\phi_m^2 y_1^2 \right.$$
$$\left. -\frac{\lambda}{2q}\phi_m y_1 \sum_\alpha y_\alpha^2 + \frac{i\lambda}{6q^2}\phi_m^3 y_1^3 + \ldots\right). \tag{12}$$

When n is close to zero the first coefficient on the right hand side is unity. We then formally integrate over y_α for $\alpha \neq 1$, so that we are left with only one integral in the limit $n \to 0$:

$$I_m(\lambda) = \int dy_1 \sqrt{\frac{\lambda \phi_m y_1}{2\pi q}} \tag{13}$$
$$\exp\left(-\frac{\lambda}{2q}\phi_m^2 y_1^2 + \frac{\lambda}{2q}\phi_m\left(\frac{i}{3q}\phi_m^2 - 1\right) y_1^3 + \ldots\right)$$

To compute this integral, we can still try to apply a saddle point method to the function inside the exponential. This function is proportional to λ and the first derivative vanishes for two solutions, $y_a = 0$ and $y_b = 2q\phi_m/(i\phi_m^2 - 3q)$. The values of the second derivatives are equal respectively to minus and plus $\lambda\phi_m^2/q$. If we expand the integral around y_a for example, we obtain

$$I_m(\lambda) = \frac{(1+i)}{\sqrt{\pi}} \left(\frac{2q}{\lambda}\right)^{1/4} \frac{1}{\phi_m} \int_0^\infty dz \sqrt{z} \exp(-z^2) + \ldots \tag{14}$$

with $\int_0^\infty dz \sqrt{z} \exp(-z^2) = \Gamma(3/4)/2 \simeq 0.612\ 708$. The density of states is finally asymptotically equal to

$$P(\lambda) = \mathrm{Im} \frac{2}{\pi} \frac{\partial}{\partial \lambda} \left\{ \exp\left(-q\lambda \log \lambda + q\lambda - q\right) \right. \tag{15}$$
$$\left. \sum_{m=-\infty}^{+\infty} I_m(\lambda) \exp\left(2i\pi q \lambda m\right) \right\}$$

The approximation (14) is not accurate because we have considered only one saddle point, but it basically shows that for large m, I_m decreases like $1/\sqrt{|m|}$, so that the series in (15) is convergent, except for the points where $q\lambda$ is an integer. At these points the series diverges, and we obtain peaks in the distribution. A more precise computation of (13) is to find a path for which the integral is convergent. A first transformation is to make the cubic term inside the exponential purely imaginary, so that the integral is defined on the real axis. This is done by setting $z = y_1/a$ such that the coefficient a satisfies for example

$$a^3(-\frac{\lambda}{2q}\phi_m + \frac{i\lambda}{6q^2}\phi_m^3) = i \tag{16}$$

The integration path is then along the direction given by the vector a. There are 3 solutions for (16), each being proportional to $\exp(2i\pi k/3)$, with $k = 0, 1, 2$. We will note them $a_{k,m}$. Now the coefficients of the quadratic term in (13) is equal to $b_{k,m} \equiv \lambda\phi_m^2 a_{k,m}^2/2q$. Therefore I_m can expressed as

$$I_m(\lambda) = \sqrt{\frac{\lambda\phi_m}{2\pi q}}\, a_{k,m}^{3/2} \exp(i\sigma\pi/4)\psi_\sigma(b_{k,m}) \tag{17}$$

with the function

$$\psi_\sigma(b_{k,m}) = \sqrt{2}\int_0^\infty dz\sqrt{z}\exp(-b_{k,m}z^2)\left(\cos z^3 + \sigma\sin z^3\right). \tag{18}$$

where $\sigma = \pm 1$ has to be determined. In order for ψ_σ to be well defined, the real part of $b_{k,m}$ should be positive. For each m, we choose k such as $\mathrm{Re}(b_{k,m}) > 0$. We have checked numerically that there is always only one solution $k(m)$ satisfying this condition. Moreover, for $\sigma = 1$ the series over m in (15) vanishes numerically, so that $\sigma = -1$ gives a finite answer. We compute numerically $\psi_\sigma(b_{k,m})$ and $I_m(\lambda)$ for every m, and plot in figure 2 the expression (15) in the region of localization above $\lambda = 1.7$, together with data from figure 1. The asymptotic curve is in agreement for $\lambda > 1.9$ with the numerical results that use direct diagonalization of random matrices over a large number of configurations. Divergences occurs each time that $q\lambda$ is proportional to an integer. Near these points the saddle point approximation may not be accurate since in the series [equation (15)] the terms are decreasing slowly with m, and we may need more correcting terms in (13). To get a further idea on how to simplify the series, we see that, for large λ or large m, $a_{k,m}$ behaves like

$$a_{k,m} \simeq \left(\frac{6q^2}{\lambda}\right)^{1/3} \frac{1}{\phi_m}\ \exp(2i\pi k/3). \tag{19}$$

This is a good approximation for $\lambda \gg \exp(\sqrt{3/4q}) \approx 1.21$ if $q = 20$ and $m = 0$. For a ratio between the two terms inside the brackets in (16) equal to $1/10$, with $m = 0$, we obtain a value $\lambda = 1.06$. In this approximation, $b_{0,m} = (9q\lambda/2)^{1/3}$ is real and positive, and independent of m, the other solutions have negative real parts. We then obtain the following behavior for large λ

$$I_m(\lambda) \approx \sqrt{\frac{3q}{\pi}}\psi_\sigma\left(\left(\frac{9q\lambda}{2}\right)^{1/3}\right)\exp(i\sigma\pi/4)\frac{1}{\phi_m} \tag{20}$$

We find that in this limit I_m is directly proportional to the inverse of ϕ_m. The argument of ϕ_m is

$$\arg(\phi_m) = \frac{1}{2}\arctan\left(\frac{\log\lambda}{2\pi m}\right) + \frac{\pi}{2}\theta(-m), \tag{21}$$

with $\arg(\phi_0) = \pi/4$, and $\theta(x)$ is 1 for $x > 0$ and zero for $x \leq 0$. An approximation of the probability distribution function is then given by

$$P(\lambda \gg 1) \approx \sqrt{\frac{6}{\pi^3}}\frac{\partial}{\partial\lambda}\left\{\frac{1}{\sqrt{|\log\lambda|}}\exp(-q\lambda\log\lambda + q\lambda - q)\right.$$
$$\left.\psi_\sigma\left(\left(\frac{9q\lambda}{2}\right)^{1/3}\right)\sum_{m=-\infty}^{\infty}\frac{\sin(2\pi q\lambda m - \arg(\phi_m) + \sigma\pi/4)}{\left(1 + 4\pi^2 m^2/\log^2\lambda\right)^{1/4}}\right\} \tag{22}$$

The term $m = 0$ gives the monotonic part of the asymptotic curve, and in order for this term not to be zero, we have to take $\sigma = -1$. we can moreover replace ψ_σ by its asymptotic value

$$\psi_\sigma\left(\left(\frac{9q\lambda}{2}\right)^{1/3}\right) \approx \frac{1}{\sqrt{2}}\Gamma(3/4)\left(\frac{2}{9q\lambda}\right)^{1/4}. \tag{23}$$

A further approximation is to replace the denominator in (22) by $(1 + \pi^2 m^2/\log^2\lambda)$. The resulting function is less accurate than the numerical integration of (13), but it should be accurate enough for very large λ. Replacing the arctan [equation (21)] by $\pm\pi/2$, the series can be summed up using standard formulas. Instead, we would like to study the behavior of the $f(t)$ for large times, which is connected to the behavior of P at small argument. Indeed, the main contribution of the integral in (3) comes from very small λ, and therefore we need the asymptotic behavior of P in this region. We make the hypothesis that the saddle point equation [equation (6)] should still be

valid for both large and small λ, since the asymptotic solutions [equation (8)] in the two cases are the conjugates of each other using the mapping $\lambda \to 1/\lambda$. We might however modify the formula [equation (22)] since ψ_σ has a different behavior for small λ

$$\psi_\sigma\left(\left(\frac{9q\lambda}{2}\right)^{1/3}\right) \approx \frac{\sqrt{\pi}}{3}(1+\sigma) - \frac{2^{2/3}}{36}\sqrt{\pi}\frac{\Gamma(7/12)}{\Gamma(11/12)}\frac{3+\sqrt{3}}{3-\sqrt{3}}(\sigma-2+\sqrt{3})\left(\frac{9q\lambda}{2}\right)^{1/3} \tag{24}$$

and we also need to define the new arguments of the complex saddle points

$$\arg(\phi_m) = \frac{1}{2}\arctan\left(\frac{\log\lambda}{2\pi m}\right) - \frac{\pi}{2}\theta(-m), \tag{25}$$

with $\arg(\phi_0) = -\pi/4$. In that case, $\sigma = 1$, so that ψ_1 is roughly constant for small λ [equation (24)]. This result show that (22) should still be an asymptotic solution for small and non zero values of λ, with ψ_1 instead of ψ_{-1}. This may explains the oscillations seen in the low eigenvalue region in figure 1 (inset), for $0.2 < \lambda < 0.5$, with $q = 20$ and $N = 800$. we can notice that, despite the fact that the solutions from (8) are conjugate by $\lambda \to 1/\lambda$, and thus possess some symmetry, the two regions $\lambda \ll 1$ and $\lambda \gg 1$ appear not to have symmetrical asymptotic distributions, since the global distribution itself inside the mobility edges is not symmetric.

A further study would be to precise the physical meaning of these peaks in the localized region, and this actually appears in the structure of the real eigenvectors (see [1]). It may correspond to the particle trap in a region of low or high connectivity (defects) as suggested by BM. An interesting model given by BM is a Cayley tree with a defect on the central site, with a connectivity c instead of $q+1$ for the other sites. They found some localized states corresponding to, for example, strong c. This localized states can disappear if a connectivity c' for the surrounding neighbors is introduced and if c' is small enough, giving rise to a *screening* effect. Also, it is not clear however why the structure is $1/q$ periodic and if the peaks observed are Dirac peaks or are diverging with a power law like our result suggests. The asymptotic result (22) is also a complement to approximations given in the BM's work, and also in a different way by reference [19] where the central limit theorem is used to computed the fixed-point function of an implicit equation giving the probability distribution for an effective Hamiltonian. One of the BM's approximations is based on a *single defect approximation*, which consists to allow fluctuations of the connectivity of a single site within an effective medium. They were able to find numerically the peaks in the localized

region and gave the value of the different weights for a given N with a good approximation. In reference [19], peaks seem not to diverge for large λ, they appear to be rounded, but they are observed with a shift to the right as noted by the author. In the small λ limit, peaks seem to be sharper, but the central limit theorem may not work in this case, even if the position of these peaks are correct. Interestingly the correct shape of the distribution in the delocalized region is found with high accuracy.

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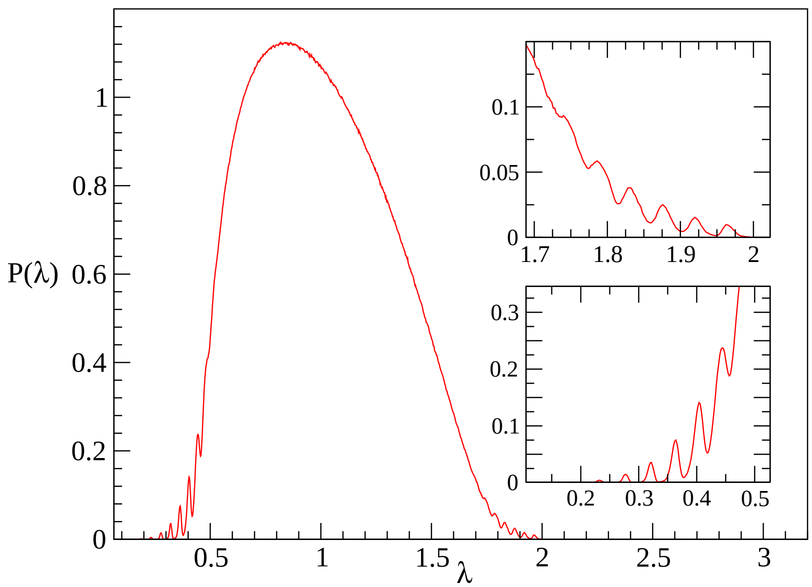

Figure 1. Probability density function $P(\lambda)$ averaged over 4500 samples with N=800 and q=20. The inserts show oscillations at the edges of the distribution.

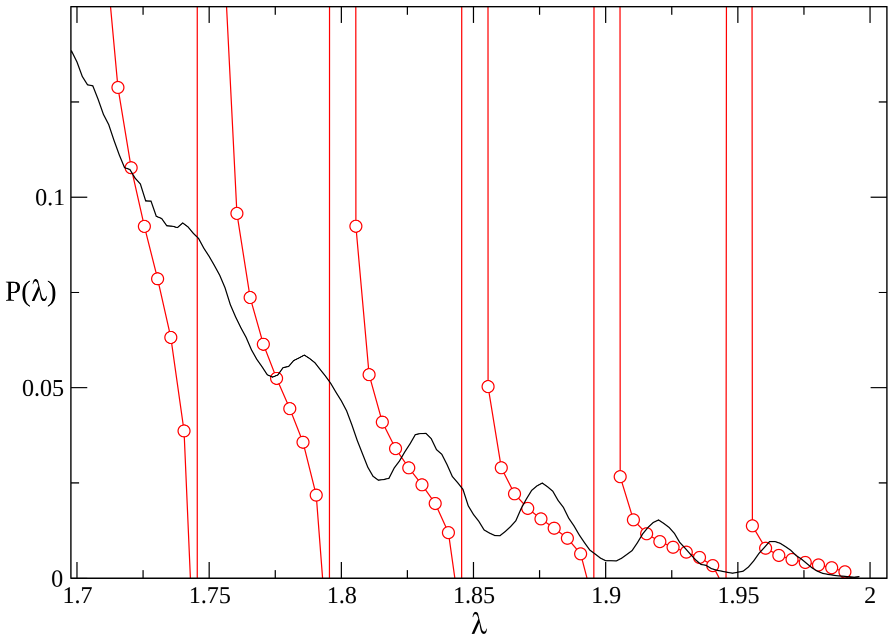

Figure 2. Probability density function $P(\lambda)$ from the asymptotic form (15) superposed with the numerical diagonalization curve from figure 1 in the large λ region.